Journal of Differential Equations **161**, 191–218 (2000)
doi:10.1006/jdeq.1999.3703, available online at http://www.idealibrary.com on IDEAL®

L_p-Convergence Rate to Nonlinear Diffusion Waves for p-System with Damping

Kenji Nishihara[1]

School of Political Science and Economics, Waseda University

Weike Wang[2]

Department of Mathematics, Wuhan University

and

Department of Mathematics, City University of Hong Kong

Received January 14, 1999; revised April 28, 1999

In this paper, we study the p-system with frictional damping and show that the solutions time-asymptotically tend to the nonlinear diffusion waves governed by the classical Darcy's law. By introducing an approximate Green function, we obtain the optimal L_p, $2 \leqslant p \leqslant +\infty$, convergence rate of the solution, which is a perturbation of the nonlinear diffusion wave, to the hyperbolic system. © 2000 Academic Press

Key Words: p-system with damping; nonlinear diffusion wave; approximate Green function; L_p estimate.

1. INTRODUCTION

In this paper, we are interested in the time-asymptotic behavior of solutions to the p-system with frictional damping

$$\begin{aligned} &v_t - u_x = 0, \\ &u_t + p(v)_x = -\alpha u, \qquad \alpha > 0, \quad p' < 0, \end{aligned} \tag{1.1}$$

[1] Supported in part by Waseda University Grant for Special Research Project 98A-504 and Grant-in-Aid for Scientific Research (C)(2) 10640216 of the Ministry of Education, Science, Sports and Culture.

[2] Supported in part by National Natural Science Foundation of China 19871065.

[3] Supported in part by the stratigic grant of City University of Hong Kong #7000729.

with the initial data

$$(v, u)(x, 0) = (v_0(x), u_0(x)).$$

Here $v(x, t) > 0$ and $u(x, t)$ represent the specific volume and velocity, respectively; the pressure $p(v)$ is assumed to be a smooth function of v with $p(v) > 0$, $p'(v) < -C_0 < 0$; and α is a positive constant. The system can be viewed as the isentropic Euler equations in Lagrangian coordinates with frictional term $-\alpha u$ in the momentum equation. (1.1) can be used to model the compressible fluid flow through a porous media.

In [5], Hsiao and Liu proved that the solutions to (1.1) time asymptotically behave like those governed by the Darcy's law in L_2 and L_∞ norms. That is, as t tends to ∞, the smooth solution $(v(x, t), u(x, t))$ which is away from vacuum approaches to the solution $(\bar{v}(x, t), \bar{u}(x, t))$ governed by the following system with the same initial data:

$$\begin{aligned} \bar{v}_t &= -\frac{1}{\alpha}\, p(\bar{v})_{xx}, \\ p(\bar{v})_x &= -\alpha \bar{u}, \end{aligned} \tag{1.2}$$

or

$$\begin{aligned} \bar{v}_t - \bar{u}_x &= 0, \\ p(\bar{v})_x &= -\alpha \bar{u}. \end{aligned} \tag{1.2$'$}$$

Recently in [13], Nishihara investigated the same problem and improved the convergence rates in L_2 and L_∞ norms by using the energy method. By combining the energy method and the pointwise estimation, for the special case, i.e, $v_0(-\infty) = v_0(\infty)$, L_∞ convergence rates obtained in [13] are optimal. This case is easier because the Green function is the heat kernel and the exact expression of the function $V(x, t)$ can be obtained. Here the function $V(x, t)$ will be introduced in Section 2.

For the system (1.1), the existence of BV solutions when the end states at $x = \pm\infty$ are the same was proved in [1]. Large data compell us to treat BV solutions. In this framework it will be interesting to consider the same problem. For the 3×3 system, i.e. (1.1) with another equation for conservation of energy, the similar problem was also studied, see [7] and references therein. For the case when vacuum appears, there is no general theory. However, Liu in [8] gave a family of special solutions to (1.1) connecting to vaccum which tend to the Barenblatt solutions time asymptotically. For other results related to the system (1.1), see [3, 4, 6, 11, 13, 14, 15] and references therein.

The main purpose of this paper is to obtain a sharper result on the convergence rates in L_p, $2 \leqslant p \leqslant \infty$. These estimates are obtained by using the method introduced by Liu as in [9] which depends on the careful study of some approximate Green function, and the energy method. By introducing an appropriate approximate Green function for the reduced equation and using some estimates obtained from energy estimation, we give the optimal L_p, $2 \leqslant p \leqslant \infty$, convergence rates for the genaral cases.

The rest of the paper is arranged as follows: The main theorem is stated in Section 2; In Section 3, we restate some properties of the diffusion wave of (1.2) and some energy estimates obtained in [13]; In Section 4, the approximate Green function is introduced. The proof of the main theorem is given in the last section.

Throughout this paper we denote the generic constants by C. $W^{m,p}(\mathbf{R})$, $m \in \mathbf{Z}_+$, $p \in [2, \infty]$, denotes the usual Sobolev space with its norm

$$\|f\|_{W^{m,p}} := \sum_{k=0}^{m} \|\partial_x^k f\|_{L_p}.$$

In particular, $W^{m,2} = H^m$ and $\|\cdot\| = \|\cdot\|_{L_2}$. Moreover, the domain $\mathbf{R}$ will be abbreviated without any ambiguity.

2. THE MAIN RESULT

We are interested in the behavior of the solution of (1.1) with initial data satisfying

$$(v, u)(x, 0) \to (v_\pm, u_\pm) \qquad \text{as} \quad x \to \pm\infty, \tag{2.1}$$

with v_+ not necessarily equal to v_-. Denote the self-similar solution, diffusion wave, of (1.2) in the form of $\varphi(x/\sqrt{t+1})$ by $\bar{v}(x, t)$ with the same end states as $v(x, 0)$:

$$\bar{v}(\pm\infty, 0) = v_\pm, \tag{2.2}$$

and set

$$\bar{u}(x, t) \equiv -\frac{1}{\alpha}\, p(\bar{v})_x. \tag{2.3}$$

Since the u component of the solution is expected to decay exponentially at $x = \pm\infty$, as in [5], the auxiliary functions are needed to eliminate the u values at $x = \pm\infty$. The functions $\tilde{u}$ and $\tilde{v}$ are the solution of

$$\begin{aligned} \tilde{v}_t - \tilde{u}_x &= 0 \\ \tilde{u}_t &= -\alpha\tilde{u}. \end{aligned} \tag{2.4}$$

with $\tilde{u}(x, t) \to e^{-\alpha t} u_{\pm}$ as $x \to \pm\infty$. For definiteness, we choose them as those in [5]:

$$\tilde{u}(x, t) = e^{-\alpha t}\left(u_- + (u_+ - u_-) \int_{-\infty}^{x} m_0(y)\, dy\right) \tag{2.5}$$

and

$$\tilde{v}(x, t) = \frac{u_+ - u_-}{-\alpha}\, e^{-\alpha t} m_0(x), \tag{2.6}$$

where $m_0(x)$ is a smooth function with compact support satisfying

$$\int_{-\infty}^{\infty} m_0(x)\, dx = 1.$$

Let the initial data $v_0(x)$ be a small perturbation of a diffusion wave, and we are going to study how the solution behaves as t tends to ∞. As shown in [5] and [13], there exists a shift x_0 of $\bar{v}(x, t)$ such that at time zero,

$$\int_{-\infty}^{\infty} (v_0(y) - \bar{v}(y + x_0, 0) - \tilde{v}(y, 0))\, dy = 0.$$

By using the first equation of (1.1), it is easy to show that the function $V(x, t)$ defined by

$$V(x, t) = \int_{-\infty}^{x} (v(y, t) - \bar{v}(y + x_0, t) - \tilde{v}(y, t))\, dy \tag{2.7}$$

satisfies $V(\pm\infty, t) = 0$. Here x_0 is a constant uniquely determined by

$$\int_{-\infty}^{\infty} (v(x, 0) - \bar{v}(x + x_0, 0))\, dx = \frac{u_+ - u_-}{-\alpha}. \tag{2.8}$$

For later use, we denote

$$U(x, t) = u(x, t) - \bar{u}(x + x_0, t) - \tilde{u}(x, t), \tag{2.9}$$

and $V_0(x) = V(x, 0)$, $U_0(x) = U(x, 0) = V_t(x, 0)$. From (1.1), (1.2), (2.4), (2.7), (2.8), and (2.9), we can reformulate (1.1) as

$$\begin{aligned} &V_t - U = 0, \\ &U_t + (p(V_x + \bar{v} + \tilde{v}) - p(\bar{v}))_x + \alpha U = \frac{1}{\alpha}\, p(\bar{v})_{xt}, \\ &(V, U)|_{t=0} \equiv (V_0, U_0)(x) \to 0 \qquad \text{as} \quad x \to \pm\infty. \end{aligned} \tag{2.10}$$

By linearizing the second equation of (2.10) about $\bar{v}$, we have the following linearized system

$$
\begin{aligned}
&V_t - U = 0, \\
&U_t + (p'(\bar{v})\, V_x)_x + \alpha U = F_1 + F_2, \qquad\qquad (2.11)\\
&(V, U)|_{t=0} \equiv (V_0, U_0)(x) \to 0 \qquad \text{as} \quad x \to \pm\infty,
\end{aligned}
$$

where $F_j(x, t) = (\tilde{F}_j(x, t))_x$ $(j = 1, 2)$, and

$$
\begin{aligned}
&\tilde{F}_1(x, t) = \frac{1}{\alpha}\, p(\bar{v})_t, \qquad\qquad (2.12)\\
&\tilde{F}_2(x, t) = -(p(V_x + \bar{v} + \tilde{v}) - p(\bar{v}) - p'(\bar{v})\, V_x).
\end{aligned}
$$

From now on, we will study the system (2.11). For the completeness of the paper and the comparison of the convergence rates, we list the main theorems in [5] and [13] as follows.

THEOREM 2.1 (Hsiao and Liu, [5]). *If* $V_0(x) = V(x, 0) \in H^3(\mathbf{R})$, $U_0(x) = V_t(x, 0) \in H^2(\mathbf{R})$, *and*

$$|u_+ - u_-| + |v_+ - v_-| + \|V_0\|_{H^3} + \|U_0\|_{H^2} \leqslant \varepsilon_0$$

for some sufficiently small ε_0, *then there exists a global in time solution* $V(x, t) \in L^\infty([0, \infty), H^3(\mathbf{R}))$, $U(x, t) \in L^\infty([0, \infty), H^2(\mathbf{R}))$ *of* (2.10), *which satisfies*

$$\|(V_x, U)(t)\|_{L_2} + \|(V_x, U)(t)\|_{L_\infty} = O(1)\, \varepsilon_0 (1+t)^{-1/2}. \qquad (2.13)$$

THEOREM 2.2 (Nishihara [13]). *Under the conditions of Theorem* 2.1, *there exists a global in time solution of* (2.10) *which satisfies*

$$V(x, t) \in W^{\bar{k}, \infty}([0, \infty); H^{3-\bar{k}}),\ U(x, t) \in W^{k, \infty}([0, \infty); H^{2-k}) \qquad (2.14)$$

for $\bar{k} = 0, 1, 2, 3$; $k = 0, 1, 2$, *and*

$$
\begin{aligned}
&\|\partial_x^k V_x(t)\|_{L_2} = O(1)\, \varepsilon_0 (1+t)^{-(k+1)/2},\ \|\partial_x^k U(t)\|_{L_2} \\
&\qquad = O(1)\, \varepsilon_0 (1+t)^{-(k+2)/2}, \qquad\qquad (2.15)\\
&\|V_x(t)\|_{L_\infty} = O(1)\, \varepsilon_0 (1+t)^{-3/4},\ \|U(t)\|_{L_\infty} = O(1)\, \varepsilon_0 (1+t)^{-5/4}. \qquad (2.16)
\end{aligned}
$$

Moreover, if $v_+ = v_-$, $u_+ = u_- = 0$ *and* $V_0, U_0 \in L_1(\mathbf{R})$ *with*

$$\int_{-\infty}^{\infty} (v_0(x) - v_-)\, dx = 0,$$

then

$$\|V_x(t)\|_{L_\infty} = O(1)(1+t)^{-1}, \qquad \|U(t)\|_{L_\infty} = O(1)(1+t)^{-3/2}. \quad (2.17)$$

Our main result of this paper is the following.

THEOREM 2.3. *If* $V_0(x) \in H^3(\mathbf{R}) \cap L_1(\mathbf{R})$, $U_0(x) \in H^2(\mathbf{R}) \cap L_1(\mathbf{R})$, *and*

$$|u_+ - u_-| + |v_+ - v_-| + \|V_0\|_{H^3} + \|U_0\|_{H^2} + \|V_0\|_{L_1} + \|U_0\|_{L_1} \leqslant \varepsilon_0$$

for some sufficiently small ε_0, *then there exists a global in time solution* $V(x, t)$, $U(x, t)$ *of* (2.10), *which satisfies* (2.14) *and*

$$\|\partial_x^k V_x(t)\|_{L_p} = O(1)\,\varepsilon_0(1+t)^{-(1-1/p)/2-(k+1)/2}, \quad (2.18)$$

$$\|\partial_x^k U(t)\|_{L_p} = O(1)\,\varepsilon_0(1+t)^{-(1-1/p)/2-k/2-1} \quad (2.19)$$

for any $k \leqslant 2$ *if* $p = 2$ *and* $k \leqslant 1$ *if* $p \in (2, +\infty]$.

Our result shows that the decay rates (2.18) and (2.19) with $k = 0$ and $p = \infty$ correspond to (2.17) even in the case $v_+ \neq v_-$, which are sharper than those in [13].

3. SOME KNOWN ESTIMATES

In this section we will restate some known properties of the nonlinear diffusion wave $\bar{v}$ to (1.2) and (2.1) and the L_2 estimates of the derivatives of the function $V(x, t)$ (see (2.7)), all of which were obtained in, or follow from [5] and [13]. We list them here for the convenience of the readers.

First, we list the propeties of $\bar{v}(x, t)$ as follows: The function $\bar{v}(x, t)$ possesses the form

$$\begin{aligned} &\bar{v}(x, t) = \varphi(x/\sqrt{1+t}) \equiv \varphi(\xi), \qquad -\infty < \xi < +\infty, \\ &\varphi(\pm\infty) = v_\pm. \end{aligned} \quad (3.1)$$

Moreover, the function $\varphi(\xi)$ satisfies

$$\sum_{k=1}^{6} |\varphi^{(k)}(\xi)| + |\varphi(\xi) - v_+|_{\xi>0} + |\varphi(\xi) - v_-|_{\xi<0} \leqslant C\,|v_+ - v_-|\,e^{-C\alpha\xi^2}, \quad (3.2)$$

where $\varphi^{(k)}(\xi)$ denotes the derivative of $\varphi(\xi)$ of kth order with respect to ξ, and C is a positive constant. According to the form of the function $\bar{v}(x, t)$, we have

$$\begin{aligned}
&\bar{v}_x=\frac{\varphi'(\xi)}{\sqrt{t+1}}, \quad \bar{v}_t=-\frac{\xi\varphi'(\xi)}{2(t+1)}, \quad \bar{v}_{xx}=\frac{\varphi''(\xi)}{t+1}, \quad \bar{v}_{xt}=\frac{\varphi'(\xi)+\xi\varphi''(\xi)}{2(t+1)\sqrt{t+1}},\\
&\bar{v}_{tt}=\frac{3\xi\varphi'(\xi)+\xi^2\varphi''(\xi)}{4(t+1)^2}, \quad \bar{v}_{xxx}=\frac{\varphi'''(\xi)}{(t+1)\sqrt{t+1}},\\
&\bar{v}_{xtt}=\frac{3\varphi'(\xi)+5\xi\varphi''(\xi)+\xi^2\varphi'''(\xi)}{4(t+1)^2\sqrt{t+1}}\\
&\bar{v}_{xttt}=-\frac{15\varphi'(\xi)+33\xi\varphi''(\xi)+12\xi^2\varphi'''(\xi)+\xi^3\varphi''''(\xi)}{8(t+1)^3\sqrt{t+1}}
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
\int |\bar{v}_x(x, t)|^2\, dx &= O(1)\, |v_+-v_-|^2\, (t+1)^{-1/2},\\
\int (|\bar{v}_t|^2+|\bar{v}_{xx}|^2)\, dx &= O(1)\, |v_+-v_-|^2\, (t+1)^{-3/2},\\
\int (|\bar{v}_{xt}|^2+|\bar{v}_{xxx}|^2)\, dx &= O(1)\, |v_+-v_-|^2\, (t+1)^{-5/2},\\
\int |\bar{v}_{tt}|^2\, dx &= O(1)\, |v_+-v_-|^2\, (t+1)^{-7/2},\\
\int |\bar{v}_{xtt}|^2\, dx &= O(1)\, |v_+-v_-|^2\, (t+1)^{-9/2}\\
\int |\bar{v}_{xttt}|^2\, dx &= O(1)\, |v_+-v_-|^2\, (t+1)^{-13/2}.
\end{aligned} \tag{3.4}$$

Next we will give an L_2 estimate on U_{tt} and U_{tx} which will be used later. Before doing this, we restate the following lemma from [13].

LEMMA 3.1. *Suppose that both* $\delta := |v_+-v_-|+|u_+-u_-|$ *and* $\|V_0\|_3+\|U_0\|_2$ *are sufficiently small. Then, there exists a unique global in time solution* $(V, U)(x, t)$ *of* (2.10), *which satisfies*

$$V\in W^{i,\infty}([0, \infty); H^{3-i}), \quad i=0, 1, 2, 3,$$

and moreover

$$
\begin{aligned}
&\sum_{k=0}^{3} (1+t)^k \, \|\partial_x^k V(\cdot, t)\|^2 \\
&\qquad + \sum_{k=0}^{2} (1+t)^{k+2} \, \|\partial_x^k U(\cdot, t)\|^2 + (1+t)^3 \, \|\partial_t U(\cdot, t)\|^2 \\
&\qquad + \int_0^t \left[\sum_{j=1}^{3} (1+\tau)^{j-1} \, \|\partial_x^j V(\cdot, \tau)\|^2 + \sum_{j=0}^{2} (1+\tau)^{j+1} \, \|\partial_x^j U(\cdot, \tau)\|^2 \right] d\tau \\
&\quad \leqslant C(\|V_0\|_3^2 + \|U_0\|_2^2 + \delta).
\end{aligned}
$$

Similar to the proof of the above Lemma in [13], we have the following estimate on U_{tt} after tedious calculations. This estimate for the case $v_- = v_+$ was also obtained in [13].

LEMMA 3.2. *Under the hypothesis of Lemma* 3.1, *we have the following estimate for the decay rate of* $\|U_{tt}\|^2$.

$$
\begin{aligned}
&(t+1)^5 \, (\|U_{tt}\|^2 + \|U_{xt}\|^2) + \int_0^t (\tau+1)^5 \, \|U_{tt}\|^2 \, d\tau \\
&\qquad \leqslant C(\|V_0\|_3^2 + \|U_0\|_2^2 + \delta). \qquad (3.6)
\end{aligned}
$$

Outline of the Proof. Differentiate the second equation of (2.11) with respect to t twice and multiply it by U_{tt}. By using the properties of the diffusion wave and Lemma 3.1, after some calculation we have

$$
\begin{aligned}
&\frac{d}{dt} \int_{-\infty}^{\infty} (U_{tt}^2 - p'(\bar{v}) \, U_{xt}^2) \, dx + \alpha \int_{-\infty}^{\infty} U_{tt}^2 \, dx \\
&\quad \leqslant C\varepsilon_0 \left[(t+1)^{-1} \int_{-\infty}^{\infty} U_{xt}^2 \, dx + (t+1)^{-2} \int_{-\infty}^{\infty} U_{xx}^2 \, dx \right. \\
&\qquad \left. + (t+1)^{-3} \int_{-\infty}^{\infty} U_x^2 \, dx \right] \\
&\qquad + C\delta \left[(t+1)^{-4} \int_{-\infty}^{\infty} V_{xx}^2 \, dx + (t+1)^{-5} \int_{-\infty}^{\infty} V_x^2 \, dx + (t+1)^{-13/2} \right].
\end{aligned}
$$

Multiplying the above inequality by $(t+1)^5$ and integrate from 0 to t, and using the Lemma 3.1 again yields the estimate (3.6). ∎

4. GREEN FUNCTION

We rewrite (2.11) as

$$\alpha V_t - (a(x, t)\, V_x)_x = F - V_{tt}, \tag{4.1}$$

where $a(x, t) = -p'(\bar{v}(x, t)) > C_0 > 0$, $F = F_1 + F_2$. Now we will construct an approximate Green function $G(x, t;\, y, s)$ for (4.1) which is continuous and piecewise smooth. It satisfies the basic requirement

$$G(x, t;\, y, t) = \delta(y - x), \tag{4.2}$$

where δ is the Dirac delta function. Multiplying (4.1) whose variables are changed to (y, s) by G and integrating over the region $(y, s) \in \mathbf{R} \times (0, t)$, (4.2) gives

$$\begin{aligned} V(x, t) = {} & \int_{-\infty}^{\infty} G(x, t;\, y, 0)\, V_0(y)\, dy \\ & + \alpha^{-1} \int_0^t \int_{-\infty}^{\infty} G(x, t;\, y, s)(F(y, s) - V_{ss}(y, s))\, dy\, ds \\ & + \int_0^t \int_{-\infty}^{\infty} (G_s(x, t;\, y, s) + \alpha^{-1}(a(y, s)\, G_y(x, t;\, y, s))_y) \\ & \times V(y, s)\, dy\, ds. \end{aligned} \tag{4.3}$$

If $a(y, s)$ is a constant and G is a Green function of (4.1), we know that the last integral of (4.3) is equal to zero. But it is difficult to give an explicit expression of the Green function, such that the last integral of (4.3) is equal to zero. However, we only need to minimize the expression $\alpha G_s + (aG_y)_y$. For this purpose, we choose the following approximate Green function of (4.3):

$$G(x, t;\, y, s) = \left(\frac{\alpha}{4\pi a(x, t)(t - s)} \right)^{1/2} \exp\left(\frac{-\alpha(x - y)^2}{4A(y, s, t)(t - s)} \right), \tag{4.4}$$

where $A(y, s, t) = -p'(\varphi(\eta))$, φ is defined in (3.1), and

$$\eta = \begin{cases} y/\sqrt{1 + s}, & s > t/2, \\ y/\sqrt{1 + t/2}, & s \leqslant t/2. \end{cases}$$

It is clear that the Green function in (4.4) satisfies the condition (4.2). Setting

$$G_D(y, s) = \left(\frac{\alpha}{4\pi C_0 s}\right)^{1/2} \exp\left(\frac{-\alpha y^2}{Ds}\right), \tag{4.5}$$

for any positive constant $D > 4 \max A(y, s, t) + O(1)\,\varepsilon$. Denote $\theta(t, s) = \theta_1(t, s) + \theta_2(t, s)$ with

$$\begin{aligned} \theta_1(t, s) &= \begin{cases} (1+s)^{-1/2}, & s > t/2, \\ 0, & s \leqslant t/2, \end{cases} \\ \theta_2(t, s) &= \begin{cases} 0, & s > t/2, \\ (1+t)^{-1/2}, & s \leqslant t/2. \end{cases} \end{aligned} \tag{4.6}$$

Using above notations, if $l \leqslant 1$, $h \leqslant 1$, we have

$$\begin{aligned} &|\partial_t^l \partial_s^h \partial_x^k \partial_y^m G(x, t; y, s)| \\ &\quad = O(1) \left(\sum_{m_1 + m_2 = m} (t-s)^{(-m_1 - k)/2} \theta^{m_2} \right) \\ &\quad\quad \times (\theta_2^2 + (t-s)^{-1})^l \, (\theta_1^2 + (t-s)^{-1})^h \, G_D(x-y, t-s). \end{aligned} \tag{4.7}$$

For heat kernel $G_D(x, t)$, we have

$$\|G_D(\cdot, t)\|_{L_p} \leqslant C t^{-1/2(1-1/p)}. \tag{4.8}$$

Since the function $G(x, t; y, s)$ is not symmetric with respect to variables x and y, t and s, we need the following formulas:

$$\partial_x G = -\partial_y G - G\left(\frac{\alpha(x-y)^2}{4A^2(t-s)}\, A'_y + \frac{a'_x(x, t)}{2a(x, t)}\right),$$

and

$$\partial_t G = -\partial_s G - G\left(\frac{\alpha(x-y)^2}{4A^2(t-s)}\, (A'_s + A'_t) - \frac{a'_t(x, t)}{2a(x, t)}\right).$$

It follows from (3.2) that

$$|A'_y| + |a'_x| = O(1)\,\varepsilon_0 \theta, \qquad |A'_s| + |A'_t| + |a'_t| = O(1)\,\varepsilon_0 \theta^2,$$

where A'_y, A'_t and A'_s represent the derivative of A with respect to y, t, s respectively. Given a function $g(y, s)$ and two constants $0 \leqslant a < b \leqslant t$, we have

$$
\begin{aligned}
&\int_a^b \int_{-\infty}^{\infty} \partial_x G(x,t;y,s)\, g(y,s)\, dy\, ds \\
&\qquad = \int_a^b \int_{-\infty}^{\infty} G(x,t;y,s)\, \partial_y\, g(y,s)\, dy\, ds \\
&\qquad\quad + O(1)\, \varepsilon_0 \int_a^b \int_{-\infty}^{\infty} \theta(t,s)\, G_D(x-y,t-s)\, g(y,s)\, dy\, ds,
\end{aligned} \tag{4.9}
$$

and

$$
\begin{aligned}
&\int_a^b \int_{-\infty}^{\infty} \partial_t G(x,t;y,s)\, g(y,s)\, dy\, ds \\
&\qquad = \int_a^b \int_{-\infty}^{\infty} G(x,t;y,s)\, \partial_s\, g(y,s)\, dy\, ds \\
&\qquad\quad - \int_{-\infty}^{\infty} G(x,t;y,s)\, g(y,s)\, dy \Big|_{s=a}^{s=b} \\
&\qquad\quad + O(1)\, \varepsilon_0 \int_a^b \int_{-\infty}^{\infty} \theta^2(t,s)\, G_D(x-y,t-s)\, g(y,s)\, dy\, ds.
\end{aligned} \tag{4.10}
$$

In general, for $k \geqslant 1$, we have

$$
\begin{aligned}
&\int_a^b \int_{-\infty}^{\infty} \partial_x^{h+k} G(x,t;y,s)\, g(y,s)\, dy\, ds \\
&\qquad = \int_a^b \int_{-\infty}^{\infty} \partial_x^h G(x,t;y,s)\, \partial_y^k\, g(y,s)\, dy\, ds \\
&\qquad\quad + O(1)\, \varepsilon_0 \sum_{\beta<k} \int_a^b \int_{-\infty}^{\infty} (t-s)^{-h/2}\, \theta^{(k-\beta)}(t,s)\, G_D\, \partial_y^\beta\, g(y,s)\, dy\, ds,
\end{aligned} \tag{4.11}
$$

and

$$
\begin{aligned}
&\int_a^b \int_{-\infty}^{\infty} \partial_t \partial_x^{h+k} G(x,t;y,s)\, g(y,s)\, dy\, ds \\
&\qquad = \int_a^b \int_{-\infty}^{\infty} \partial_x^h G(x,t;y,s)\, \partial_s \partial_y^k\, g(y,s)\, dy\, ds \\
&\qquad\quad - \int_{-\infty}^{\infty} \partial_x^{h+k} G(x,t;y,s)\, g(y,s)\, dy \Big|_{s=a}^{s=b} \\
&\qquad\quad + O(1)\, \varepsilon_0 \sum_{\beta<k} \int_a^b \int_{-\infty}^{\infty} (t-s)^{-h/2}\, \theta^{2+(k-\beta)}(t,s)\, G_D\, \partial_y^\beta\, g(y,s)\, dy\, ds.
\end{aligned} \tag{4.12}
$$

Similarly, we have

$$\int_{t/2}^{t}\int_{-\infty}^{\infty} \partial_x^3 G(x,t;y,s)\, g(y,s)\, dy\, ds$$
$$= \int_{t/2}^{t}\int_{-\infty}^{\infty} \partial_y^2 G(x,t;y,s)\, \partial_y g(y,s)\, dy\, ds$$
$$+ O(1)\,\varepsilon_0 \sum_{\beta\leqslant 1} \int_{t/2}^{t}\int_{-\infty}^{\infty} ((t-s)^{-1/2}+\theta)\, \theta^{(2-\beta)}(t,s)\, G_D\, \partial_y^\beta\, g(y,s)\, dy\, ds. \tag{4.13}$$

Denoting

$$R_G \equiv G_s(x,t;y,s) + \alpha^{-1}(a(y,s)\, G_y(x,t;y,s))_y,$$

the first term of the right hand of (4.13) can be rewritten as

$$\int_{t/2}^{t}\int_{-\infty}^{\infty} \partial_y^2 G(x,t;y,s)\, \partial_y g(y,s)\, dy\, ds$$
$$= \int_{t/2}^{t}\int_{-\infty}^{\infty} \left(\alpha a^{-1}(y,s)(R_G - \partial_s G) - \frac{\partial_y a}{a}(y,s)\, \partial_y G\right) \partial_y g(y,s)\, dy\, ds.$$

Using integration by part of variables y and s, we have

$$\int_{t/2}^{t}\int_{-\infty}^{\infty} a^{-1}(y,s)\, \partial_s G(x,t;y,s)\, \partial_y g(y,s)\, dy\, ds$$
$$= \int_{t/2}^{t}\int_{-\infty}^{\infty} a^{-1}(y,s)\, \partial_y G\, \partial_s g(y,s)\, dy\, ds$$
$$+ \int_{-\infty}^{\infty} a^{-1}(y,s)\, G\, \partial_y g(y,s)\, dy\Big|_{s=t/2}^{s=t}$$
$$+ O(1)\,\varepsilon_0 \int_{t/2}^{t}\int_{-\infty}^{\infty} G_D(t-s, x-y)(\theta^2\, \partial_y g(y,s) + \theta\, \partial_s g(y,s))\, dy\, ds.$$

With these estimates, we can get from (4.13) that

$$\int_{t/2}^{t}\int_{-\infty}^{\infty} \partial_x^3 G(x,t;y,s)\, g(y,s)\, dy\, ds$$
$$= \int_{t/2}^{t}\int_{-\infty}^{\infty} \left(\alpha a^{-1}(y,s)\, R_G - \frac{\partial_y a}{a}(y,s)\, \partial_y G\right) \partial_y g(y,s)\, dy\, ds$$
$$- \alpha \int_{t/2}^{t}\int_{-\infty}^{\infty} a^{-1}(y,s)\, \partial_y G\, \partial_s g(y,s)\, dy\, ds$$

$$-\alpha \int_{-\infty}^{\infty} a^{-1}(y, s)\, G\, \partial_y g(y, s)\, dy\Big|_{s=t/2}^{s=t}$$

$$+ O(1)\, \varepsilon_0 \sum_{\beta \leqslant 1} \int_{t/2}^{t} \int_{-\infty}^{\infty} ((t-s)^{-1/2} + \theta)\, \theta^{(2-\beta)}(t, s)\, G_D\, \partial_y^{\beta}\, g(y, s)\, dy\, ds$$

$$+ O(1)\, \varepsilon_0 \int_{t/2}^{t} \int_{-\infty}^{\infty} \theta(t, s)\, G_D\, \partial_s\, g(y, s)\, dy\, ds. \tag{4.14}$$

Similarly, we can conclude that

$$\int_{t/2}^{t} \int_{-\infty}^{\infty} \partial_t \partial_x^2\, G(x, t;\, y, s)\, g(y, s)\, dy\, ds$$

$$= \int_{t/2}^{t} \int_{-\infty}^{\infty} \partial_y^2\, G(x, t;\, y, s)\, \partial_s\, g(y, s)\, dy\, ds - \int_{-\infty}^{\infty} \partial_x^2\, G g(y, s)\, dy\Big|_{s=t/2}^{s=t}$$

$$+ O(1)\, \varepsilon_0 \sum_{\beta \leqslant 1} \int_{t/2}^{t} \int_{-\infty}^{\infty} ((t-s)^{-1/2} + \theta)\, \theta^{(3-\beta)}(t, s)\, G_D\, \partial_y^{\beta}\, g(y, s)\, dy\, ds.$$

Letting $g(y, s) = \partial_y\, \tilde{g}(y, s)$ and using the same method, we have

$$\int_{t/2}^{t} \int_{-\infty}^{\infty} \partial_t \partial_x^2\, G(x, t;\, y, s)\, \partial_y\, \tilde{g}(y, s)\, dy\, ds$$

$$= \int_{t/2}^{t} \int_{-\infty}^{\infty} \left(\alpha a^{-1}(y, s)\, R_G - \frac{\partial_y a}{a}\, (y, s)\, \partial_y G \right) \partial_s \partial_y\, \tilde{g}(y, s)\, dy\, ds$$

$$-\alpha \int_{t/2}^{t} \int_{-\infty}^{\infty} a^{-1}(y, s)\, \partial_y G\, \partial_s^2\, \tilde{g}(y, s)\, dy\, ds$$

$$- \int_{-\infty}^{\infty} (\partial_y^2 G\, \partial_y\, \tilde{g}(y, s) + \alpha a^{-1}(y, s)\, G\, \partial_s \partial_y\, \tilde{g}(y, s))\, dy\Big|_{s=t/2}^{s=t}$$

$$+ O(1)\, \varepsilon_0 \sum_{\beta \leqslant 1} \int_{t/2}^{t} \int_{-\infty}^{\infty} ((t-s)^{-1/2} + \theta)\, \theta^{(3-\beta)}(t, s)\, G_D\, \partial_y^{\beta+1} \tilde{g}(y, s)\, dy\, ds$$

$$+ O(1)\, \varepsilon_0 \int_{t/2}^{t} \int_{-\infty}^{\infty} (\theta(t, s)\, G_D\, \partial_s^2 + \theta^2(t, s)\, G_D\, \partial_s \partial_y)\, \tilde{g}(y, s)\, dy\, ds. \tag{4.15}$$

We next estimate $R_G \equiv G_s + \alpha^{-1}(a G_y)_y$. After some calculations, we have

$$R_G = O(1)\, \varepsilon_0\, \Theta(t, s)\, \tilde{E}(y, t, s)\, G_D(x-y, t-s), \tag{4.16}$$

where

$$\Theta(t, s) = \begin{cases} ((1+s)^{-1} + (t-s)^{-1/2}(1+s)^{-1/2}), & s > t/2 \\ ((1+t)^{-1} + (t-s)^{-1/2}(1+s)^{-1/2}), & s \leqslant t/2, \end{cases}$$

$$\tilde{E}(y, t, s) = \begin{cases} E(y, s), & s > t/2, \\ E(y, t), & s \leqslant t/2, \end{cases}$$

with $E(y, \tau) = \exp(-C\alpha y^2/(1+\tau))$. If $s < t/2$, by direct calculation, we know

$$\begin{aligned} &|\partial_t^l \, \partial_x^k R_G(x, t; y, s)| \\ &\quad \leqslant C\varepsilon_0 (1+s)^{-1/2} \, (1+t)^{-(l+(k+1)/2)} \, E(y, t) \, G_D(x-y, t-s). \end{aligned} \tag{4.17}$$

Notice that $R_G(x, t; y, s)$ is discontinuous at $s = t/2$. When $s = t/2$, we have

$$\begin{aligned} &\lim_{s \to t/2\pm} |\partial_x^k \, R_G(x, t; y, s)| \\ &\quad \leqslant C\varepsilon_0 (1+t)^{-(1+k/2)} \, E(y, t/2) \, G_D(x-y, t/2). \end{aligned} \tag{4.18}$$

Moreover, for a given function $g(y, s)$, we have

$$\begin{aligned} &\int_{t/2}^{t} \int_{-\infty}^{\infty} \partial_x^k \, R_G(x, t; y, s) \, g(y, s) \, dy \, ds \\ &= \int_{t/2}^{t} \int_{-\infty}^{\infty} R_G(x, t; y, s) \, \partial_y^k \, g(y, s) \, dy \, ds \\ &\quad + O(1) \, \varepsilon_0 \sum_{\beta < k} \int_{t/2}^{t} \int_{-\infty}^{\infty} \Theta \theta^{(k-\beta)} E(y, s) \, G_D(x-y, t-s) \, \partial_y^\beta \, g(y, s) \, dy \, ds, \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} &\int_{t/2}^{t} \int_{-\infty}^{\infty} \partial_t \partial_x^k \, R_G(x, t; y, s) \, g(y, s) \, dy \, ds \\ &= \int_{t/2}^{t} \int_{-\infty}^{\infty} R_G(x, t; y, s) \, \partial_s \partial_y^k \, g(y, s) \, dy \, ds \\ &\quad + \int_{-\infty}^{\infty} \partial_x^k \, R_G(x, t; y, s) \, g(y, s) \, dy \Big|_{s=t/2}^{s=t} \\ &\quad + O(1) \, \varepsilon_0 \sum_{\beta < k} \int_{t/2}^{t} \int_{-\infty}^{\infty} \Theta \theta^{2+(k-\beta)} E(y, s) \\ &\quad \times G_D(x-y, t-s) \, \partial_y^\beta \, g(y, s) \, dy \, ds. \end{aligned} \tag{4.20}$$

5. THE PROOF OF MAIN THEOREM

In this section, we will give some estimates on $\partial_t^l \partial_x^k V(x,t)$, $(l \leqslant 1, k+l \leqslant 3)$ by using the approximate Green function. Denoting

$$\begin{aligned}
I_1^{l,k} &= \int_{-\infty}^{\infty} \partial_t^l \partial_x^k G(x,t;y,0)\, V(y,0)\, dy,\\
I_2^{l,k} &= \partial_t^l \int_0^t \int_{-\infty}^{\infty} \partial_x^k G(x,t;y,s)\, F_1(y,s)\, dy\, ds,\\
I_3^{l,k} &= \partial_t^l \int_0^t \int_{-\infty}^{\infty} \partial_x^k G(x,t;y,s)\, F_2(y,s)\, dy\, ds, \qquad (5.1)\\
I_4^{l,k} &= \partial_t^l \int_0^t \int_{-\infty}^{\infty} \partial_x^k G(x,t;y,s)(V_{ss}(y,s))\, dy\, ds,\\
I_5^{l,k} &= \partial_t^l \int_0^t \int_{-\infty}^{\infty} \partial_x^k R_G(x,t;y,s)\, V(y,s)\, dy\, ds,
\end{aligned}$$

from (4.3) we can write

$$\partial_t^l \partial_x^k V(x,t) = I_1^{l,k} + \alpha^{-1} I_2^{l,k} + \alpha^{-1} I_3^{l,k} - \alpha^{-1} I_4^{l,k} + I_5^{l,k}. \qquad (5.2)$$

Set

$$B_p^{l,k}(t) = (1+t)^{1/2(1-1/p)+l+k/2}, \qquad (5.3)$$

and

$$\begin{aligned}
M(t) = &\sup_{p>2,\, 0\leqslant s\leqslant t,\, l+k\leqslant 2,\, l\leqslant 1} B_p^{l,k}(s)\, \|\partial_t^l \partial_x^k V(\cdot,s)\|_{L_p}\\
&+ \sup_{0\leqslant s\leqslant t,\, l+k=3,\, l\leqslant 1} B_p^{l,k}(s)\, \|\partial_t^l \partial_x^k V(\cdot,s)\|_{L_2}. \qquad (5.4)
\end{aligned}$$

We are going to estimate $I_j^{l,k}$ as follows. Since $\|V_0\|_{L_1} \leqslant C\varepsilon_0$, $\|U_0\|_{L_1} \equiv \|V_s(\cdot,0)\|_{L_1} \leqslant C\varepsilon_0$, (4.7), (4.8) and Hausdorff–Young inequality gives

$$\|I_1^{l,k}\|_{L_p} \leqslant C\varepsilon_0 \|(1+t)^{-l-k/2}\, G_D(\cdot,t)\|_{L_p} \leqslant C\varepsilon_0 (1+t)^{-1/2(1-1/p)-l-k/2}. \qquad (5.5)$$

For $I_2^{l,k}$, it is easy to see that from (3.1) and (3.2)

$$\|\partial_t^l \partial_x^k \bar{v}\|_{L_p} \leqslant C\varepsilon_0 (1+t)^{-l-k/2+1/2p}, \qquad (5.6)$$

provided that $1 \leqslant l+k \leqslant 6$. Thus for $l+k \leqslant 5$, we have

$$\|\partial_t^l \partial_x^k \tilde{F}_1\|_{L_p} \leqslant C\varepsilon_0 (1+t)^{-l-k/2-1/2-1/2(1-1/p)}. \tag{5.7}$$

By (4.11) and integration by part with respect to variable y, we have

$$\begin{aligned}\|I_2^{0,k}\|_{L_p} \leqslant C\Bigg(&\int_{t/2}^t \sum_{h\leqslant k} \\ &\times \left\| \int_{-\infty}^{\infty} (1+s)^{-(k-h)/2}\, G_D(\cdot - y, t-s)\, \partial_y^h F_1(y,s)\, dy \right\|_{L_p} ds \\ &+ \int_0^{t/2} \left\| \int_{-\infty}^{\infty} \partial_y \partial_x^k G(\cdot, t; y, s)\, \tilde{F}_1(y,s)\, dy \right\|_{L_p} ds \Bigg).\end{aligned}$$

Thus, (4.7) and (5.7) give

$$\begin{aligned}\|I_2^{0,k}\|_{L_p} \leqslant C &\int_{t/2}^t ((t-s)^{-1/2(1-1/p)}\, \varepsilon_0 (1+s)^{-1-k/2})\, ds \\ &+ \int_0^{t/2} (1+t)^{-1/2(1-1/p)-k+1/2}\, \varepsilon_0 (1+s)^{-1/2})\, ds \\ &\leqslant C\varepsilon_0 (1+t)^{-1/2(1-1/p)-k/2}.\end{aligned}$$

If $l=1$, then (4.12) yields

$$\begin{aligned}\|I_2^{1,k}\|_{L_p} \leqslant C\Bigg(&\int_{t/2}^t \sum_{h\leqslant k,\, m\leqslant 1} \\ &\times \left\| \int_{-\infty}^{\infty} (1+s)^{-(k-h)/2-(1-m)}\, G_D\, \partial_s^m \partial_y^h\, F_1(y,s)\, dy \right\|_{L_p} ds \\ &+ \int_0^{t/2} \left\| \int_{-\infty}^{\infty} \partial_t \partial_y \partial_x^k\, G(\cdot, t; y, s)\, \tilde{F}_1(y,s)\, dy \right\|_{L_p} ds \\ &+ \left\| \int_{-\infty}^{\infty} \partial_x^k G(\cdot, t; y, t/2)\, F_1(y, t/2)\, dy \right\|_{L_p} \Bigg).\end{aligned}$$

The estimation of the first and the second terms of the above equality is similar to those of $I_2^{0,k}$. The last term can be estimated by (4.7) and (5.7):

$$\left\| \int_{-\infty}^{\infty} \partial_x^k G(\cdot, t; y, t/2)\, F_1(y, t/2)\, dy \right\|_{L_p} \leqslant C\varepsilon_0 (1+t)^{-1/2(1-1/p)-k/2-1}.$$

Thus, we obtain

$$\|I_2^{l,k}\|_{L_p} \leqslant C\varepsilon_0(1+t)^{-1/2(1-1/p)-k/2-l}. \tag{5.8}$$

Next, we first consider $I_5^{l,k}$. When $l=0, k\leqslant 3$, by (4.19) we have

$$I_5^{0,k} = \int_0^{t/2}\int_{-\infty}^{\infty} \partial_x^k\, R_G \cdot V\, dy\, ds + \int_{t/2}^{t}\int_{-\infty}^{\infty} R_G \cdot \partial_y^k V\, dy\, ds$$

$$+ O(1)\,\varepsilon_0 \sum_{\beta<k} \int_{t/2}^{t}\int_{-\infty}^{\infty} (1+s)^{-(k-\beta)}\, \Theta E(y,s)\, G_D \cdot \partial_y^\beta V\, dy\, ds.$$

Using (4.16), (4.17), Hausdorff–Young inequality and Hölder inequality, we have

$$\|I_5^{0,k}\|_{L_p} \leqslant C\varepsilon_0 \Bigg(\int_0^{t/2} (1+s)^{-1/2}(1+t)^{-(k+1)/2}\, \|G_D\|_{L_p}\, \|E(\cdot,t)\|_{L_2}\, \|V\|_{L_2}\, ds$$

$$+ \int_{t/2}^{t} \Theta\, \|G_D\|_{L_2}\, \|E(\cdot,s)\|_{L_p}\, \|\partial_y^k V\|_{L_2}\, ds$$

$$+ \sum_{\beta<k}\int_{t/2}^{t} (1+s)^{-(k-\beta)/2}\, \Theta\, \|G_D\|_{L_2}\, \|E(\cdot,s)\|_{L_2}\, \|\partial_y^\beta V\|_{L_p}\, ds \Bigg)$$

$$\leqslant C\varepsilon_0 \Bigg(\int_0^{t/2} (1+t)^{-(k+1)/2}\, (t-s)^{-1/2(1-1/p)}$$

$$\times (1+t)^{1/4}\, (1+s)^{-1/2(1-1/2)-1/2}\, ds$$

$$+ \sum_{\beta\leqslant k}\int_{t/2}^{t} (1+s)^{-(k-\beta)/2}\, \Theta(t-s)^{-1/2(1-1/2)}$$

$$\times (1+s)^{1/2p}\, (1+s)^{-1/2(1-1/2)-\beta/2}\, ds \Bigg).$$

Since

$$\sum_{\beta\leqslant k}\int_{t/2}^{t} (1+s)^{-(k-\beta)/2}\, \Theta(t-s)^{-1/2(1-1/2)}\, (1+s)^{1/2p}\, (1+s)^{-1/2(1-1/2)-\beta/2}\, ds$$

$$\leqslant = C\int_{t/2}^{t} (1+s)^{-1/2(1-1/p)-k/2+1/4}\, (t-s)^{-1/4}$$

$$\times ((1+s)^{-1} + (t-s)^{-1/2}\, (1+s)^{-1/2})\, ds$$

$$\leqslant C(1+t)^{-1/2(1-1/p)-k/2},$$

we have

$$\|I_5^{0,k}\| \leqslant C\varepsilon_0(1+t)^{-1/2(1-1/p)-k/2}.$$

If $l=1$, $k \leqslant 2$, we have

$$\begin{aligned}
I_5^{1,k} &= \partial_t \left(\left(\int_{t/2}^{t} + \int_0^{t/2}\right) \int_{-\infty}^{\infty} \partial_x^k R_G(x, t; y, s)\, V(y, s)\, dy\, ds\right) \\
&= \int_{t/2}^{t} \int_{-\infty}^{\infty} (\partial_t \partial_x^k R_G)\, V\, dy\, ds + \int_0^{t/2} \int_{-\infty}^{\infty} (\partial_t \partial_x^k R_G)\, V\, dy\, ds \\
&\quad + \int_{-\infty}^{\infty} (\partial_x^k R_G)\, V\, dy\,|_{s=t} - \lim_{s \to t/2+} \int_{-\infty}^{\infty} (\partial_x^k R_G)\, V\, dy \\
&\quad + \lim_{s \to t/2-} \int_{-\infty}^{\infty} (\partial_x^k R_G)\, V\, dy.
\end{aligned}$$

Equations (4.20), (4.17) and (4.18) give

$$\begin{aligned}
\|I_5^{1,k}\|_{L_p} \leqslant C\varepsilon_0 \Bigg(& \int_0^{t/2} (1+s)^{-1/2} (1+t)^{-(k+3)/2} \|G_D\|_{L_p} \|E(\cdot, t)\|_{L_2} \|V\|_{L_2}\, ds \\
&+ \sum_{\beta \leqslant k,\, \alpha \leqslant 1} \int_{t/2}^{t} (1+s)^{-(k-\beta)/2-(1-\alpha)} \\
&\times \Theta\, \|G_D\|_{L_2} \|E(\cdot, s)\|_{L_p} \|\partial_s^\alpha \partial_y^\beta V\|_{L_2}\, ds \\
&+ (1+t)^{-1-k/2} \|G_D(\cdot, t/2)\|_{L_p} \|E(\cdot, t/2)\|_{L_2} \|\partial_y^\beta V(\cdot, t/2)\|_{L_2} \Bigg).
\end{aligned}$$

By the similar method as the one for the case of $l=0$, we can get

$$\|I_5^{1,k}\|_{L_p} \leqslant C\varepsilon_0(1+t)^{-1/2(1-1/p)-k/2-1}.$$

Thus for $l+k \leqslant 3$ and $l \leqslant 1$, we have

$$\|I_5^{l,k}\|_{L_p} \leqslant C\varepsilon_0(1+t)^{-1/2(1-1/p)-k/2-l}. \tag{5.9}$$

We now turn to $I_3^{l,k}$. First we can write

$$\tilde{F}_2 = -\left(p'(\bar{v})\, \tilde{v} + p''(\bar{v} + \mu(V_x + \tilde{v}))((V_x + \tilde{v})^2/2\right),$$

with $0 < \mu < 1$. Since $m_0(x)$ is a smooth function with compact support and

$$|\partial_t^l\, \partial_x^k \tilde{v}(x, t)| \leqslant C\, |u_+ - u_-|\, e^{-\alpha t}\, \partial_x^k m_0(x),$$

we have

$$|\partial_t^l \partial_x^k \tilde{v}(x,t)| \leqslant C\varepsilon_0 e^{-\alpha t} e^{-x^2}. \tag{5.10}$$

Using Hölder inequality, Lemmas 3.1, 3.2, (5.3) and (5.4), we have

$$\|\partial_t^l \partial_x^k ((p''(\bar{v}+\mu(\tilde{v}+V_x))\, V_x^2)\|_{L_p} \leqslant CM^2(t)(1+t)^{-1/2(1-1/p)-(3+k)/2-l}. \tag{5.11}$$

Since we have estimates only on $V(y,s) \in W^{m,\infty}([0,\infty); H^{3-m})$ for $m \leqslant 3$, (5.11) holds only when $l+k \leqslant 2$ and $l \leqslant 1$. When $l+k \leqslant 2$ and $l \leqslant 1$, (5.6), (5.10) and (5.11) yield

$$\|\partial_t^l \partial_x^k \tilde{F}_2\|_{L_p} \leqslant C(\varepsilon_0 + M^2(t))(1+t)^{-1/2(1-1/p)-l-(3+k)/2}; \tag{5.12}$$

When $l=2, k=0$, (5.4), Lemma 3.2 and Hölder inequality give

$$\|\partial_t^2 ((p''(\bar{v}+\theta(\tilde{v}+V_x))\, V_x^2)\|_{L_p} \leqslant C(\varepsilon_0 + M^2(t))(1+t)^{-1/2(1-1/q)-5/2-1/2}, \tag{5.13}$$

where q is a constant satisfying $1/q+1/2=1/p+1$. Thus, from (5.6), (5.10) and (5.13), we have

$$\|\partial_t^2 \tilde{F}_2\|_{L_p} \leqslant C(\varepsilon_0 + M^2(t))(1+t)^{-1/2(1-1/p)+1/4-3}. \tag{5.14}$$

We now consider $I_3^{l,k}$. When $k \leqslant 2$ and $l=0$, it follows from (4.11) that

$$\begin{aligned}
\|I_3^{0,k}\|_{L_p} \leqslant C\Bigg(&\int_{t/2}^{t} \left\| \int_{-\infty}^{\infty} \partial_x G(\cdot,t;y,s)\, \partial_y^{k-1} F_2(y,s)\, dy \right\|_{L_p} ds \\
&+ \int_0^{t/2} \left\| \int_{-\infty}^{\infty} \partial_x^k G(\cdot,t;y,s)\, F_2(y,s)\, dy \right\|_{L_p} ds\Bigg) \\
&+ C\varepsilon_0 \int_{t/2}^{t} \left\| \int_{-\infty}^{\infty} \theta(t,s)\, G_D F_2(y,s)\, dy \right\|_{L_p} ds,
\end{aligned}$$

by Hausdorff–Young inequality, (4.7) and (5.12), we have

$$\begin{aligned}
\|I_3^{0,k}\|_{L_p} &\leqslant C(\varepsilon_0 + M^2(t))\Bigg(\int_0^{t/2} (1+t)^{-1/2(1-1/p)-k/2}(1+s)^{-2}\, ds \\
&\quad + \int_{t/2}^{t} ((t-s)^{-1/2} + (1+s)^{-1/2})(1+s)^{-1/2(1-1/p)-2-k-1/2}\, ds\Bigg) \\
&\leqslant C(\varepsilon_0 + M^2(t))(1+t)^{-1/2(1-1/p)-k/2}.
\end{aligned}$$

When $k \leqslant 1$ and $l = 1$, it follows from (4.12) that

$$\|I_3^{1,k}\|_{L_p} \leqslant C\bigg(\int_{t/2}^{t} \left\| \int_{-\infty}^{\infty} \partial_x^k\, G(\cdot, t; y, s)\, \partial_s F_2(y, s)\, dy \right\|_{L_p} ds$$

$$+ \int_0^{t/2} \left\| \int_{-\infty}^{\infty} \partial_s\, \partial_x^k\, G(\cdot, t; y, s)\, F_2(y, s)\, dy \right\|_{L_p} ds$$

$$+ O(1)\, \varepsilon_0 \int_{t/2}^{t} \int_{-\infty}^{\infty} \theta^2(t, s)\, G_D F_2(y, s)\, dy\, ds$$

$$+ \left\| \int_{-\infty}^{\infty} \partial_x^k\, G(\cdot, t; y, t/2)\, F_2(y, t/2)\, dy| \right\|_{L_p}.$$

Equations (4.7) and (5.12) give

$$\|I_3^{1,k}\|_{L_p} \leqslant C(\varepsilon_0 + M^2(t))(1+t)^{-1/2(1-1/p)-1-k/2}.$$

For $l + k = 3$, since $(t-s)^{-h}$ is non-integrable in $[t, t/2]$ if $h \geqslant 1$ and $V(y, s) \in W^{m, \infty}([0, \infty); H^{3-m})$ for $m \leqslant 3$, we must replace $\partial_x^2 G(x, t; y, s)$ by $R_G(x, t; y, s)$ and $\partial_s G(x, t; y, s)$ by using (4.14) and (4.15). In fact, it follows from (4.14) that

$$\|I_3^{0,3}\|_{L_p} \leqslant \int_0^{t/2} \left\| \int_{-\infty}^{\infty} \partial_x^3 G(\cdot, t; y, s)\, F_2(y, s)\, dy \right\|_{L_p} ds$$

$$+ C \int_{t/2}^{t} \Bigg(\left\| \int_{-\infty}^{\infty} R_G\, \partial_y F_2(y, s)\, dy \right\|_{L_p} + \left\| \int_{-\infty}^{\infty} \partial_y\, G \partial_s F_2(y, s)\, dy \right\|_{L_p}$$

$$+ \left\| \int_{-\infty}^{\infty} (1+s)^{-1/2}\, \partial_y G\, \partial_y F_2(y, s)\, dy \right\|_{L_p} \Bigg) ds$$

$$+ C \left\| \int_{-\infty}^{\infty} G\, \partial_y F_2(y, s)\, dy|_{s=t/2}^{s=t} \right\|_{L_p}$$

$$+ C\varepsilon_0 \sum_{\beta \leqslant 1} \int_{t/2}^{t}$$

$$\left\| \int_{-\infty}^{\infty} ((t-s)^{-1/2} + \theta)\, \theta^{(2-\beta)}(t, s)\, G_D\, \partial_y^{\beta} F_2(y, s)\, dy \right\|_{L_p} ds$$

$$+ C\varepsilon_0 \int_{t/2}^{t} \left\| \int_{-\infty}^{\infty} \theta(t, s)\, G_D\, \partial_s F_2(y, s)\, dy \right\|_{L_p} ds.$$

Then, (4.7), (5.9) and (5.12) yield

$$\|I_3^{0,3}\|_{L_p} \leqslant C(\varepsilon_0 + M^2(t))(1+t)^{-1/2(1-1/p)-3/2}.$$

We now estimate $I_3^{1,2}$ by using (4.15)

$$\begin{aligned}
\|I_3^{1,2}\|_{L_p} \leqslant & \int_0^{t/2} \left\| \int_{-\infty}^{\infty} \partial_t \partial_x^2 \, G(\cdot, t; y, s) \, F_2(y, s) \, dy \right\|_{L_p} ds \\
& + C \int_{t/2}^{t} \left(\left\| \int_{-\infty}^{\infty} R_G \, \partial_s \partial_y \tilde{F}_2(y, s) \, dy \right\|_{L_p} \right. \\
& + \left\| \int_{-\infty}^{\infty} \partial_y G \, \partial_s^2 \tilde{F}_2(y, s) \, dy \right\|_{L_p} \\
& \left. + \left\| \int_{-\infty}^{\infty} (1+s)^{-1/2} \, \partial_y G \, \partial_s \partial_y \tilde{F}_2(y, s) \, dy \right\|_{L_p} \right) ds \\
& + C \left\| \int_{-\infty}^{\infty} (G \, \partial_s \partial_y + \partial_x^2 G \, \partial_y) \, \tilde{F}_2(y, s) \, dy|_{s=t/2} \right\|_{L_p} \\
& + C \left\| \int_{-\infty}^{\infty} G \, \partial_s \partial_y \tilde{F}_2(y, s) \, dy|_{s=t} \right\|_{L_p} \\
& + C\varepsilon_0 \sum_{\beta \leqslant 1} \int_{t/2}^{t} \left\| \int_{-\infty}^{\infty} ((t-s)^{-1/2} + \theta) \right. \\
& \left. \times \theta^{(3-\beta)}(t, s) \, G_D \, \partial_y^{\beta+1} \tilde{F}_2(y, s) \, dy \right\|_{L_p} ds \\
& + C\varepsilon_0 \int_{t/2}^{t} \\
& \left\| \int_{-\infty}^{\infty} (\theta^2 G_D \, \partial_s \partial_y + \theta G_D \, \partial_s^2) \, \tilde{F}_2(y, s) \, dy \right\|_{L_p} ds.
\end{aligned}$$

The proof is very similar to the proof of $I_3^{0,3}$, by noticing that we can use (5.14) not (5.12) for $\partial_s^2 \tilde{F}_2(y, s)$. It follows that

$$\|I_3^{1,2}\|_{L_p} \leqslant C(\varepsilon_0 + M^2(t))(1+t)^{-1/2(1-1/p)-2}.$$

In summary, for $l+k \leqslant 3$ and $l \leqslant 1$, we have

$$\|I_3^{l,k}\|_{L_p} \leqslant C(\varepsilon_0 + M^2(t))(1+t)^{-1/2(1-1/p)-k/2-l}. \tag{5.15}$$

Finally, we consider $I_4^{l,k}$. When $l=0$, $k \leqslant 2$, by (4.11) and intergration by part to variable s, we have

$$\|I_4^{0,k}\|_{L_p} \leqslant C\left(\int_{t/2}^{t}\left\|\int_{-\infty}^{\infty} \partial_x^{k-1} G(\cdot, t; y, s)\, \partial_y \partial_s^2 V(y, s)\, dy\right\|_{L_p} ds\right.$$

$$+\int_{t/2}^{t}\left\|\int_{-\infty}^{\infty} (1+s)^{-1/2}\, G_D(\cdot - y, t-s)\, \partial_s^2 V(y, s)\, dy\right\|_{L_p} ds$$

$$+\int_{0}^{t/2}\left\|\int_{-\infty}^{\infty} \partial_x^k\, \partial_s G(\cdot, t; y, s)\, \partial_s V(y, s)\, dy\right\|_{L_p} ds$$

$$\left.+\left\|\int_{-\infty}^{\infty} \partial_x^k G(\cdot, t; y, s)\, \partial_s V(y, s)\, dy|_{s=0}^{s=t/2}\right\|_{L_p}\right).$$

Let q be a constant satisfying $1/q + 1/2 = 1/p + 1$. By (4.7), Lemmas 3.1 and 3.2, we have

$$\|I_4^{0,k}\|_{L_p} \leqslant C\varepsilon_0\left(\int_{t/2}^{t}(t-s)^{-1/2(1-1/q)-k-1/2}((1+s)^{-52}+(1+s)^{-2})\, ds\right.$$

$$+\int_{0}^{t/2}(t-s)^{-1/2(1-1/q)-k/2-1}(1+s)^{-1}\, ds$$

$$\left.+(1+t)^{-1/2(1-1/q)-k/2-1}\right)+\|U_0\|_{L_1}(1+t)^{-1/2(1-1/p)-k/2}$$

$$\leqslant C\varepsilon_0(1+t)^{-1/2(1-1/p)-k/2}.$$

If $l=1$, $k \leqslant 1$, by (4.12) and intergration by part to variable s, we have

$$\|I_4^{1,k}\|_{L_p} \leqslant C\left(\int_{t/2}^{t}\left\|\int_{-\infty}^{\infty} \partial_x^k G(\cdot, t; y, s)\, \partial_s^3 V(y, s)\, dy\right\|_{L_p} ds\right.$$

$$+\int_{t/2}^{t}\left\|\int_{-\infty}^{\infty} (1+s)^{-1}\, G_D(\cdot - y, t-s)\, \partial_s^2 V(y, s)\, dy\right\|_{L_p} ds$$

$$+\int_{0}^{t/2}\left\|\int_{-\infty}^{\infty} \partial_x^k\, \partial_s^2 G(\cdot, t; y, s)\, \partial_s V(y, s)\, dy\right\|_{L_p} ds$$

$$+\left\|\int_{-\infty}^{\infty} \partial_s \partial_x^k G(\cdot, t; y, s)\, \partial_s V(y, s)\, dy|_{s=0}^{s=t/2}\right\|_{L_p}$$

$$\left.+\left\|\int_{-\infty}^{\infty} \partial_x^k G(\cdot, t; y, s)\, \partial_s^2 V(y, s)\, dy|_{s=t/2}\right\|_{L_p}\right).$$

Similarly, we have

$$\|I_4^{1,k}\|_{L_p} \leqslant C\varepsilon_0 \left(\int_{t/2}^{t} (t-s)^{-1/2(1-1/q)-k/2} (1+s)^{-5/2}\, ds \right.$$

$$+ \int_0^{t/2} (t-s)^{-1/2(1-1/q)-k/2-2} (1+s)^{-1}\, ds$$

$$\left. + (1+t)^{-1/2(1-1/q)-k/2-3/2} \right) + \|U_0\|_{L_1} (1+t)^{-1/2(1-1/p)-k/2-1}$$

$$\leqslant C\varepsilon_0 (1+t)^{-1/2(1-1/p)-k/2-1}.$$

For $I_4^{0,3}$, we also need to replace $\partial_x^2 G(x, t; y, s)$ by $R_G(x, t; y, s)$ and $\partial_s G(x, t; y, s)$. It follows from (4.14) that

$$\|I_4^{0,3}\|_{L_p} \leqslant \int_0^{t/2} \left\| \int_{-\infty}^{\infty} \partial_x^3 G(\cdot, t; y, s)\, V_{ss}(y, s)\, dy \right\|_{L_p} ds$$

$$+ C \int_{t/2}^{t} \left(\left\| \int_{-\infty}^{\infty} R_G\, \partial_y V_{ss}(y, s)\, dy \right\|_{L_p} \right.$$

$$+ \left\| \int_{-\infty}^{\infty} \partial_y G\, \partial_s V_{ss}(y, s)\, dy \right\|_{L_p}$$

$$+ \left\| \int_{-\infty}^{\infty} (1+s)^{-1/2}\, \partial_y G\, \partial_y V_{ss}(y, s)\, dy \right\|_{L_p} \Bigg)\, ds$$

$$+ C \left\| \int_{-\infty}^{\infty} G\, \partial_y V_{ss}(y, s)\, dy \Big|_{s=t/2}^{s=t} \right\|_{L_p}$$

$$+ C\varepsilon_0 \sum_{\beta \leqslant 1} \int_{t/2}^{t}$$

$$\left\| \int_{-\infty}^{\infty} ((t-s)^{-1/2} + \theta)\, \theta^{(2-\beta)}(t, s)\, G_D\, \partial_y^{\beta} V_{ss}(y, s)\, dy \right\|_{L_p} ds$$

$$+ C\varepsilon_0 \int_{t/2}^{t} \left\| \int_{-\infty}^{\infty} \theta(t, s)\, G_D\, \partial_s V_{ss}(y, s)\, dy \right\|_{L_p} ds.$$

Since

$$\int_{-\infty}^{\infty} G\, \partial_y V_{ss}(y, s)\, dy \big|_{s=t} = V_{xtt}(x, t),$$

we have from Lemma 3.2 that

$$\left\| \int_{-\infty}^{\infty} G\, \partial_y V_{ss}(y, s)\, dy|_{s=t} \right\|_{L_2} \leqslant C\varepsilon_0 (1+t)^{-5/2}.$$

Using Hausdorff–Young inequality, (4.7), (4.16), Lemmas 3.1 and 3.2, we have

$$\|I_4^{0,3}\|_{L_2} \leqslant C\varepsilon_0 (1+t)^{-12(1-12)-32}.$$

For $l=1, k=2$, we denote $I_4^{1,2} = I_{4,1}^{1,2} + I_{4,2}^{1,2}$, where

$$I_{4,1}^{1,2} = \int_0^{t-1} \int_{-\infty}^{\infty} \partial_t \partial_x^2\, G(x, t; y, s)(V_{ss}(y, s))\, dy\, ds$$

$$+ \int_{-\infty}^{\infty} \partial_x^2\, G(x, t; y, s)\, V_{ss}(y, s)\, dy|_{s=t-1},$$

$$I_{4,2}^{1,2} = \int_{t-1}^{t} \int_{-\infty}^{\infty} \partial_t \partial_x^2\, G(x, t; y, s)(V_{ss}(y, s))\, dy\, ds$$

$$- \int_{-\infty}^{\infty} \partial_x^2\, G(x, t; y, s)\, V_{ss}(y, s)\, dy|_{s=t-1}$$

$$+ \partial_x^2\, \partial_t^2\, V(x, t).$$

We first consider $I_{4,1}^{1,2}$. Using (4.12) and integrating by part of the variable s,

$$\|I_{4,1}^{1,2}\|_{L_p} \leqslant C \Bigg(\int_0^{t/2} \left\| \int_{-\infty}^{\infty} \partial_x^2\, \partial_t \partial_s\, G(\cdot, t; y, s)\, \partial_s V(y, s)\, dy \right\|_{L_p} ds$$

$$+ \left\| \int_{-\infty}^{\infty} \partial_x^2\, \partial_t G(\cdot, t; y, s)\, \partial_s V(y, s)\, dy|_{s=0}^{s=t/2} \right\|_{L_p}$$

$$+ \int_{t/2}^{t-1} \left\| \int_{-\infty}^{\infty} \partial_x^2 G(\cdot, t; y, s)\, \partial_s^3 V(y, s)\, dy \right\|_{L_p} ds$$

$$+ \int_{t/2}^{t-1} \Bigg\| \int_{-\infty}^{\infty} (1+s)^{-1}(t-s)^{-1}$$

$$\times G_D(\cdot - y, t-s)\, \partial_s^2 V(y, s)\, dy \Bigg\|_{L_p} ds$$

$$+ \left\| \int_{-\infty}^{\infty} \partial_x^2 G(\cdot, t; y, t/2)\, \partial_s^2 V(y, t/2)\, dy \right\|_{L_p} \Bigg).$$

Using Hausdoff–Young inequality, (4.7), Lemmas 3.1 and 3.2 again, we obtain

$$\|I^{1,\,2}_{4,\,1}\|_{L_p} \leqslant C\varepsilon_0 (1+t)^{-1/2(1-1/p)-2}.$$

Notice that we can replace $\partial_x^2 G(x, t;\, y, s)$ by $R_G(x, t;\, y, s)$ and $\partial_s G(x, t;\, y, s)$. Similar to the proof of (4.15), we have

$$\begin{aligned}
\|I^{1,\,2}_{4,\,2}\|_{L_p} \leqslant C \int_{t-1}^{t} \Bigg(& \left\| \int_{-\infty}^{\infty} R_G\, \partial_s V_{ss}(y, s)\, dy \right\|_{L_p} \\
& + \left\| \int_{-\infty}^{\infty} G\, \partial_s^2 V_{ss}(y, s)\, dy \right\|_{L_p} \\
& + \left\| \int_{-\infty}^{\infty} (1+s)^{-1/2}\, \partial_y G\, \partial_s V_{ss}(y, s)\, dy \right\|_{L_p} \Bigg)\, ds \\
& + C \left\| \int_{-\infty}^{\infty} (G\, \partial_s V_{ss}(y, s))\, dy \big|_{s=t-1}^{s=t} \right\|_{L_p} \\
& + C \left\| \int_{-\infty}^{\infty} (\partial_y^2 G - \partial_x^2 G)\, V_{ss}(y, s)\big|_{s=t-1} \right\|_{L_p} \\
& + C\varepsilon_0 \sum_{\beta \leqslant 1} \int_{t-1}^{t} \Bigg\| \int_{-\infty}^{\infty} ((t-s)^{-1/2} + \theta)\, \theta^{(3-\beta)}(t, s) \\
& \times G_D\, \partial_y^\beta V_{ss}(y, s)\, dy \Bigg\|_{L_p} ds \\
& + C\varepsilon_0 \int_{t-1}^{t} \left\| \int_{-\infty}^{\infty} \theta(t, s)\, G_D\, \partial_s V_{ss}(y, s)\, dy \right\|_{L_p} ds. \qquad (5.16)
\end{aligned}$$

As for (4.11), we have

$$\begin{aligned}
& \int_{-\infty}^{\infty} \partial_x^2 G(x, t;\, y, s)\, g(y, s)\, dy \\
& \quad = \int_{-\infty}^{\infty} \partial_y^2 G(x, t;\, y, s)\, g(y, s)\, dy \\
& \quad\quad + O(1)\, \varepsilon_0 \sum_{\beta < 2} \int_{-\infty}^{\infty} \theta^{(2-\beta)}(t, s)\, G_D\, \partial_y^\beta\, g(y, s)\, dy.
\end{aligned}$$

Thus by Lemmas 3.1 and 3.2, we have

$$
\begin{aligned}
&\left\| \int_{-\infty}^{\infty} (\partial_y^2 G - \partial_x^2 G)\, V_{ss}(y, s)|_{s=t-1} \right\|_{L_p} \\
&\quad \leqslant C\varepsilon_0 \left(\left\| \int_{-\infty}^{\infty} \theta^1(t, s)\, G_D\, \partial_y V_{ss}(y, s)\, dy|_{s=t-1} \right\|_{L_p} \right. \\
&\qquad \left. + \left\| \int_{-\infty}^{\infty} \theta^2(t, s)\, G_D V_{ss}(y, s)\, dy|_{s=t-1} \right\|_{L_p} \right. \\
&\quad \leqslant C\varepsilon_0 (1+t)^{-5/2}.
\end{aligned}
$$

Since

$$
V_{ss}(y, s) = F(y, s) - \alpha V_s(y, s) + a_y(y, s)\, V_y(y, s) + a(y, s)\, V_{yy}(y, s),
$$

the second term in (5.16) can be estimated by using the estimates for $F(y, s)$, V_s, $a_y V_s$ and $a V_{yy}$. Thus, we only need to consider the following three terms.

$$
R_1 = \int_{t-1}^{t} \int_{-\infty}^{\infty} G(x, t;\, y, s)\, V_{sss}(y, s)\, dy\, ds,
$$

$$
R_2 = \int_{t-1}^{t} \int_{-\infty}^{\infty} G_y(x, t;\, y, s)\, V_{yss}(y, s)\, dy\, ds,
$$

$$
R_3 = \int_{t-1}^{t} \int_{-\infty}^{\infty} G(x, t;\, y, s)\, \partial_s^2(a_y(y, s)\, V_y(y, s))\, dy\, ds,
$$

It follows from (4.11) that

$$
\| R_1 \|_{L_p} \leqslant C \int_{t-1}^{t} \left\| \int_{-\infty}^{\infty} G_D(t-s, \,\cdot - y)\, V_{sss}(y, s)\, dy \right\|_{L_p} ds
$$

By Hausdorff–Young inequality, (4.8) and Lemma 3.2, we have

$$
\begin{aligned}
\| R_1 \|_{L_p} &\leqslant C\varepsilon_0 \left(\int_{t-1}^{t} ((t-s)^{-1/2(1-1/q)})(1+s)^{-5/2}\, ds \right) \\
&\leqslant C\varepsilon_0 (1+t)^{-1/2(1-1/p)-2}.
\end{aligned}
$$

Similarly, we can prove that

$$
\|R_j\|_{L_p} \leqslant C\varepsilon_0 (1+t)^{-1/2(1-1/p)-2},
$$

for $j=2, 3$. Therefore, for the same reason as the proof of $I_4^{0,3}$, (4.7), (5.9), Lemmas 3.1 and 3.2 imply that

$$\|I_{4,2}^{1,2}\|_{L_2} \leqslant C\varepsilon_0(1+t)^{-1/2(1-1/2)-2}.$$

Thus we have

$$\|I_4^{l,k}\|_{L_p} \leqslant C\varepsilon_0(1+t)^{-1/2(1-1/p)-k/2-l}, \tag{5.17}$$

for $k+l \leqslant 3$ if $p=2$ and $k+l \leqslant 2$ if $p \in (2, \infty]$. Combining (5.1), (5.5), (5.8), (5.9), (5.15) and (5.17), we have the estimate

$$\|\partial_t^l \partial_x^k V(\cdot, t)\|_{L_p} \leqslant C(\varepsilon_0 + M^2(t))(1+t)^{-1/2(1-1/p)-k/2-l}. \tag{5.18}$$

Thus for $k+l \leqslant 3$ if $p=2$ and $k+l \leqslant 2$ if $p \in (2, \infty]$ we have

$$M(t) \leqslant C(\varepsilon_0 + M^2(t)).$$

By choosing ε_0 small enough and using continuity of $M(t)$ and induction, we conclude that $M(t) \leqslant C\varepsilon_0$, i.e.

$$\|\partial_t^l \partial_x^k V(\cdot, t)\|_{L_p} \leqslant C\varepsilon_0(1+t)^{-1/2(1-1/p)-k/2-l}, \tag{5.19}$$

for $k+l \leqslant 3$ if $p=2$ and $k+l \leqslant 2$ if $p \in (2, \infty]$. Since $U(x, t) = V_t(x, t)$ we have proved Theorem 2.3 from (5.19). ∎

According to the above discussion, it is easy to see that we can get the optimal estimates for higher derivatives of the solution if we know that higher derivative of initial data are small enough. In fact, we have the following theorem.

THEOREM 5.1. *If* $V_0(x) \in H^{m+1}(\mathbf{R}) \cap L_1(\mathbf{R})$, $U_0(x) \in H^m(\mathbf{R}) \cap L_1(\mathbf{R})$, *and*

$$|u_+ - u_-| + |v_+ - v_-| + \|V_0\|_{H^{m+1}} + \|U_0\|_{H^m} + \|V_0\|_{L_1} + \|U_0\|_{L_1} \leqslant \varepsilon_0$$

for some sufficiently small ε_0, *then there exists a global in time solution* $V(x, t)$, $U(x, t)$ *of* (2.10). *Moreover, if we have the following estimate*

$$\sum_{l=2}^{m} (1+t)^{2l-1} \|\partial_x^{l-2} \partial_t^2 V(\cdot, t)\|_{L_2}^2 + \sum_{l=3}^{m} (1+t)^{2l-1} \|\partial_x^{l-3} \partial_t^3 V(\cdot, t)\|_{L_2}^2 \leqslant C\varepsilon_0, \tag{5.20}$$

for $l \leqslant m$, then

$$\|\partial_x^k V_x(t)\|_{L_p} = O(1)\,\varepsilon_0(1+t)^{-(1-1/p)/2-(k+1)/2}, \tag{5.21}$$

$$\|\partial_x^k U(t)\|_{L_p} = O(1)\,\varepsilon_0(1+t)^{-(1-1/p)/2-k/2-1} \tag{5.22}$$

for any $k \leqslant m$ if $p = 2$ and $k \leqslant m-1$ if $p \in (2, \infty]$.

ACKNOWLEDGMENT

The second author wish to express his sincere gratitude to the Director of Liu Bie Ju Centre, Professor Roderick Wong, for his kind invitation and hospitality.

REFERENCES

1. C. Dafermos, A system of hyperbolic conservation laws with frictional damping, *Z. Angew. Math. Phys.* **46**, Special Issue (1995), 294–307.
2. C. T. Duyn and L. A. Van Peletier, A class of similarity solutions of the nonlinear diffusion equation, *Nonlinear Anal TMA* **1** (1977), 223–233.
3. L. Hsiao, "Quasilinear Hyperbolic Systems and Dissipative Mechanisms," World Scientific, Singapore, 1997.
4. L. Hsiao and D. Serre, Global existence of solutions for the system of compressible adiabatic flow through porous media, *SIAM J. Math. Anal.* **27** (1996), 70–77.
6. L. Hsiao and T.-P. Liu, Convergence to nonlinear diffusion waves for solutions of a system of hyperbolic conservation laws with damping, *Commun. Math. Phys.* **143** (1992), 599–605.
6. L. Hsiao and T.-P. Liu, Nonlinear diffusion phenomena of nonlinear hyperbolic system, *Chinese Ann. Math. Ser. B* **14** (1993), 465–480.
7. L. Hsiao and T. Luo, Nonlinear diffusive phenomena of solutions for the system of compressible adiabatic flow through porous media, *J. Differential Equations* **125** (1996), 329–365.
8. T.-P. Liu, Compressible flow with damping and vacuum, *Japan J. Appl. Math.* **13** (1993), 25–32.
9. T.-P. Liu, Pointwise convergence to shock waves for viscous conservation laws, *Commun. Pure Appl. Math.* **L** (1997), 1113–1182.
10. T.-P. Liu and Y. Zeng, Large time behavior of solutions general quasilinear hyperbolic-parabolic systems of conservation laws, *A.M.S. Memoirs* **599** (1997).
11. T.-P. Liu and K. Nishihara, Asymptotic behavior for scalar viscous conservation laws with boundary effect, *J. Differential Equations* **133** (1997), 296–320.
12. A. Matsumura, Global existence and asymptotics of the solutions of the second-order quasilinear hyperbolic equations with the first-order dissipation, *Publ. RIMS, Kyoto Univ.* **13** (1977), 349–379.
13. K. Nishihara, Convergence rates to nonlinear diffusion waves for solutions of system of hyperbolic conservation laws with damping, *J. Differential Equations* **131** (1996), 171–188.
14. K. Nishihara, Asymptotic behavior of solutions of quasilinear hyperbolic equations with linear damping, *J. Differential Equations* **137** (1997), 384–395.
15. K. Nishihara and T. Yang, Boundary effect on asymptotic behavior of solutions to the *p*-system with linear damping, to appear in *J. Differential Equations*.